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Domination in Bipartite Graphs

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Abstract We prove that the domination number of a graph of order n and minimum degree at least 2 that does not contain cycles of lengths 4, 5, 7, 10 or 13 is at most $\frac{3}{8}n$. Furthermore, we derive upper bounds on the domination number of bipartite graphs of given minimum degree.

Keywords domination number; cycle length; bipartite; probabilistic method

1 Introduction

The domination number $\gamma(G)$ of a (finite, undirected and simple) graph $G = (V, E)$ is the minimum cardinality of a set $D \subseteq V$ of vertices such that every vertex in $V \setminus D$ has a neighbour in D. This parameter is one of the most well-studied in graph theory and the two volume monograph [9, 10] provides an impressive account of the research related to this concept.

Fundamental results about the domination number $\gamma(G)$ are upper bounds in terms of the order n and the minimum degree δ of the graph G. Ore [14] proved that $\gamma(G) \leq \frac{n}{2}$ provided $\delta \geq 1$. For $\delta \geq 2$ and all but 7 exceptional graphs Blank [3] and McCuaig and Shepherd [13] proved

$$\gamma(G) \leq \frac{2n}{5}. \tag{1}$$

In [17] Reed proved that $\gamma(G) \leq \frac{3}{8}n$ for $\delta \geq 3$.

Bounds which are interesting for large minimum degree were obtained by Alon and Spencer [1], Arnautov [2] and Payan [15] who proved (see also Caro and Roditty [5, 6])

$$\gamma(G) \quad \leq \quad \left(\frac{1 + \ln(\delta + 1)}{\delta + 1}\right) n. \tag{2}$$

While all these bounds hold without restricting the structure of the graph, there are several partly quite recent results [4, 11, 12, 16, 18, 19] that involve conditions on the girth of the graph, i.e. the length of a shortest cycle.

In the present paper we consider the domination number of graphs of given minimum degree under different cycle conditions related to bipartite graphs. We prove a best-possible bound on the domination number of graphs of minimum degree 2 that do not contain cycles of lengths 4, 5, 7, 10 or 13 and bounds on the domination number of bipartite graphs of given minimum degree.

2 Results

Graphs as in Figure 1 show that the bound (1) [3, 13] actually remains best-possible for bipartite graphs. Therefore, it makes sense to forbid cycles of length 4. Since we are eventually interested in the domination number of bipartite graphs, we will also forbid some odd cycle lengths. Cycles of length 3 and long odd cycles can be dominated by (roughly) one third of their vertices and do not pose a problem. Therefore, it suffices to forbid some small odd cycle length at least 5. Up to the assumption on cycles of length 10 these comments motivate the hypothesis of the following result.

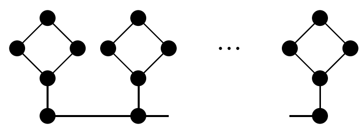

Figure 1

Theorem 1 *If* G *is a graph of order* n*, minimum degree at least* 2 *and domination number* γ *that does not contain cycles of lengths* 4*,* 5*,* 7*,* 10 *or* 13*, then* $\gamma \leq \frac{3}{8}n$*.*

Proof: For contradiction, we assume that $G = (V, E)$ is a counterexample of minimum sum of order n and size. Let n and γ be as in the statement of the theorem. Since n and γ are linear with respect to the components of G, the graph G is connected. Furthermore, the set of vertices of degree at least 3 is independent.

It is easy to check the theorem for cycles and hence we can assume that G has at least one vertex of degree at least 3.

A path between vertices of degree at least 3 with a internal vertices which are all of degree 2 is called an *a-path*. Similarly, a cycle containing a vertex of degree at least 3 and a further vertices which are all of degree 2 is called an *a-loop*. See Figure 2 for an illustration.

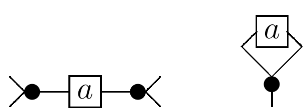

Figure 2

In what follows we will consider several times a set $V_0 \subseteq V$ of vertices with the property that $G[V \setminus V_0]$ has no vertex of degree less than 2. Note that $G[V \setminus V_0]$ satisfies the assumptions of the theorem. We will always use the following notation $n_0 = |V_0|$, $n_1 = n - n_0$, $G_0 = G[V_0]$, $G_1 = G[V \setminus V_0]$, $\gamma_0 = \gamma(G_0)$ and $\gamma_1 = \gamma(G_1)$. Note that $\gamma \leq \gamma_0 + \gamma_1$ because the union of a dominating set of G_0 and a dominating set of G_1 is a dominating set of G. Instead of a dominating set of G_0, we will sometimes consider a set $D_0 \subseteq V$ such that every vertex in V_0 is either in D_0 or adjacent to a vertex in D_0. Clearly, $\gamma \leq |D_0| + \gamma_1$.

Claim 1. *There is no a-path with $a \equiv 0$ (mod 3).*

Proof of Claim 1: For contradiction, we assume the existence of such an a-path. If V_0 is the set of internal vertices of the a-path, then $\gamma_0 = \frac{a}{3}$. By the choice of G as a minimum counterexample, we have $\gamma_1 \leq \frac{3(n-a)}{8}$ which implies the contradiction $\gamma \leq \frac{a}{3} + \frac{3(n-a)}{8} \leq \frac{3n}{8}$ and the proof of the claim is complete. □

Claim 2. *There is no a-loop with $a \equiv 0$ (mod 3).*

Proof of Claim 2: For contradiction, we assume the existence of such an a-loop containing the vertex u of degree at least 3.

If the degree of u is at least 4, we can choose V_0 as the set of the a vertices of degree 2 of the a-loop and argue as in the proof of Claim 1. Hence we can assume that the degree of u is exactly 3 and that there is a b-path leading to another vertex v of degree at least 3 (cf. Figure 3).

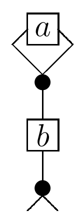

Figure 3

If V_0 consists of the $a+1$ vertices of the a-loop and the b internal vertices of the b-path, then $\gamma_0 \leq \frac{a}{3} + \left\lceil \frac{b+1}{3} \right\rceil$. Since G does not contain cycles of lengths 4, 7, 10 or 13, we have $a \geq 15$ and $b \geq 1$ which implies

$$\gamma_0 \leq \frac{a}{3} + \left\lceil \frac{b+1}{3} \right\rceil \leq \frac{a+b}{3} + 1 \leq \frac{3(a+b)}{8} + \frac{3}{8} = \frac{3n_0}{8}$$

and we obtain a similar contradiction as in the proof of Claim 1. □

Claim 3. *There is no vertex of degree at least 3 that lies on an a-loop and a b-path with $a, b \equiv 1$ (mod 3).*

Proof of Claim 3: For contradiction, we assume the existence of such a vertex u. Let the b-path lead to the vertex v of degree at least 3.

If the degree of u is 3, then let V_0 consist of the $a+1$ vertices of the a-loop and the b internal vertices of the b-path (cf. first graph in Figure 4). Clearly, $n_0 \equiv 0$ (mod 3), $\gamma_0 = \frac{n_0}{3}$ and we obtain a similar contradiction as in the proof of Claim 1.

If the degree of u is at least 5, then let V_0 consist of the a vertices of degree 2 of the a-loop and the b internal vertices of the b-path (cf. second graph in Figure 4). Now there is a set $D_0 \subseteq V$ containing u such that $|D_0| = \frac{a+b+1}{3}$ and every vertex in V_0 is either in D_0 or adjacent to a vertex in D_0. Since G does not contain cycles of lengths 5, we have $a \geq 7$ and $b \geq 1$ which implies $|D_0| \leq \frac{3n_0}{8}$. Clearly, $\gamma \leq |D_0| + \gamma_1$ and we obtain a similar contradiction as in the proof of Claim 1.

Hence we can assume that the degree of u is exactly 4 and that there is a c-path leading to a vertex w of degree at least 3. Let V_0 be a minimal set of vertices containing u for which G_1 has no vertex of degree less than 2.

Either $v \neq w$ (cf. third graph in Figure 4) or $v = w$ and v is of degree at least 4 (cf. fourth graph in Figure 4) or $v = w$, v is of degree 3 and there is a d-path leading to a vertex of degree at least 3 (cf. fifth graph in Figure 4). In all these three cases, G_0 has a spanning subgraph that arises by attaching a path with $b \geq 1$ vertices and another path with $c' \geq 1$ vertices to one vertex of a cycle of length $a + 1 \geq 8$. By the parity conditions, this implies

$$\gamma_0 \leq \frac{a+2}{3} + \frac{b-1}{3} + \left\lceil \frac{c-1}{3} \right\rceil \leq \frac{3(a+b+c+1)}{8} = \frac{3n_0}{8}$$

and we obtain a similar contradiction as in the proof of Claim 1. $\square$

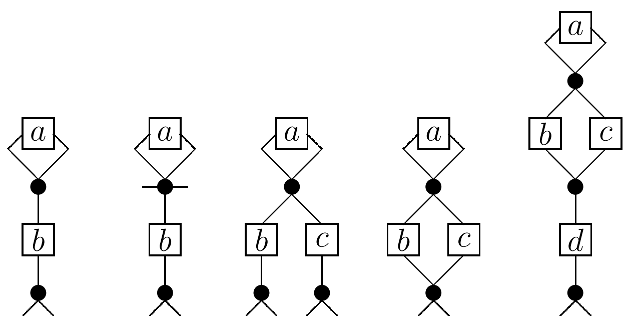

Figure 4

Claim 4. *There are no two vertices of degree at least 4 joined by an a-path and a b-path with $a, b \equiv 1$ (mod 3).*

Proof of Claim 4: For contradiction, we assume the existence of such vertices u and v. If V_0 consists of the internal vertices of the a-path and the b-path (cf. Figure 5), then there is a set $D_0 \subseteq V$ containing u such that $|D_0| = \frac{a+b+1}{3}$ and every vertex in V_0 is either in D_0 or adjacent to a vertex in D_0. Since G that does not contain cycles of lengths 4 or 7, we have $a + b \geq 8$ which implies $|D_0| \leq \frac{3n_0}{8}$. Clearly, $\gamma \leq |D_0| + \gamma_1$ and we obtain a similar contradiction as in the proof of Claim 1. $\square$

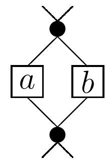

Figure 5

Claim 5. *There are no two vertices u of degree 3 and v of degree at least 4 that are joined by an a-path and a b-path with $a, b \equiv 1 \pmod 3$.*

Proof of Claim 5: For contradiction, we assume the existence of such vertices u and v. There is a c-path leading from u to a vertex w of degree at least 3. Let V_0 be a minimal set of vertices containing u for which G_1 has no vertex of degree less than 2.

Either $v \neq w$ (cf. first graph in Figure 6) or $v = w$ and v is of degree at least 5 (cf. second graph in Figure 6) or $v = w$, v is of degree 4 and there is a d-path leading from v to a vertex of degree at least 3 (cf. third graph in Figure 6). In all three cases, G_0 has a spanning subgraph that arises by attaching a path with a vertices, a path with b vertices and a path with $c' \geq 1$ vertices to a single vertex, i.e. this graph is a subdivision of a star. By the parity conditions and since $a + b \geq 8$, this implies

$$\gamma_0 \leq 1 + \frac{a-1}{3} + \frac{b-1}{3} + \left\lceil \frac{c'-1}{3} \right\rceil \leq \frac{3(a+b+c'+1)}{8} = \frac{3n_0}{8}$$

and we obtain a similar contradiction as in the proof of Claim 1. $\square$

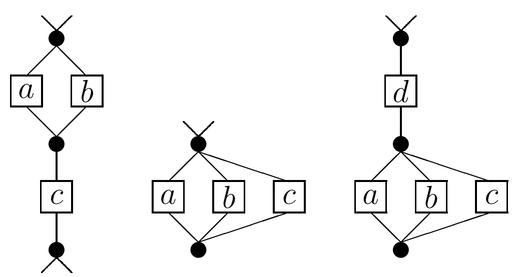

Figure 6

Claim 6. *There are no two vertices u and v of degree 3 that are joined by an a-path and a b-path and such that there is another c-path starting at u with $a, b, c \equiv 1 \pmod 3$.*

Proof of Claim 6: For contradiction, we assume the existence of such vertices u and v. Let the c-path lead from u to the vertex w of degree at least 3. There is a d-path leading from v to a vertex w' of degree at least 3. Let V_0 be a minimal set of vertices containing u for which G_1 has no vertex of degree less than 2.

By a similar reasoning as in the proofs of the previous claims, we obtain that the local structure of G is as in one of the four graphs in Figure 7. Since $a+b \geq 8$, in all these cases $\gamma_0 \leq \frac{3n_0}{8}$ and we obtain a similar contradiction as in the proof of Claim 1. (For the first three graphs we can argue exactly as in the proof of Claim 5. For the fourth graph in Figure 7 we need to use $a+b \geq 8$ and $c \equiv 1 \pmod 3$.) $\square$

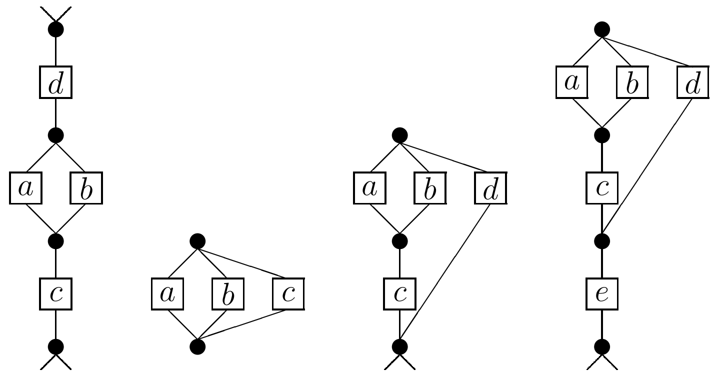

Figure 7

Claim 7. *There are no four vertices u, v_1, v_2 and v_3 of degree at least 3 such that u is joined to v_1 by an a-path, u is joined to v_2 by a b-path and u is joined to v_3 by a c-path with $a, b, c \equiv 1$ (mod 3).*

Proof of Claim 7: For contradiction, we assume the existence of such vertices. Let V_0 be a minimal set of vertices containing the internal vertices of the a-path, the b-path and the c-path for which G_1 has no vertex of degree less than 2.

By a similar reasoning as in the proofs of the previous claims, we obtain that the local structure of G is a in one of the nine graphs in Figure 8. In the first case V_0 consists of the internal vertices of the a-path, the b-path and the c-path. There is a set $D_0 \subseteq V$ containing u such that $|D_0| = \frac{a+b+c}{3} = \frac{n_0}{3}$ and every vertex in V_0 is either in D_0 or adjacent to a vertex in D_0. Again, $\gamma \leq |D_0| + \gamma_1$ and we obtain a similar contradiction as in the proof of Claim 1. In all of the remaining eight cases, G_0 has a spanning subgraph that arises by attaching a path with a vertices, a path with b vertices, a path with c vertices, and a path with $d' \geq 0$ vertices to a single vertex; i.e. this graph is a subdivision of a star. By the parity conditions, this implies $\gamma_0 \leq \frac{n_0}{3}$ and we obtain a similar contradiction as in the proof of Claim 1. $\square$

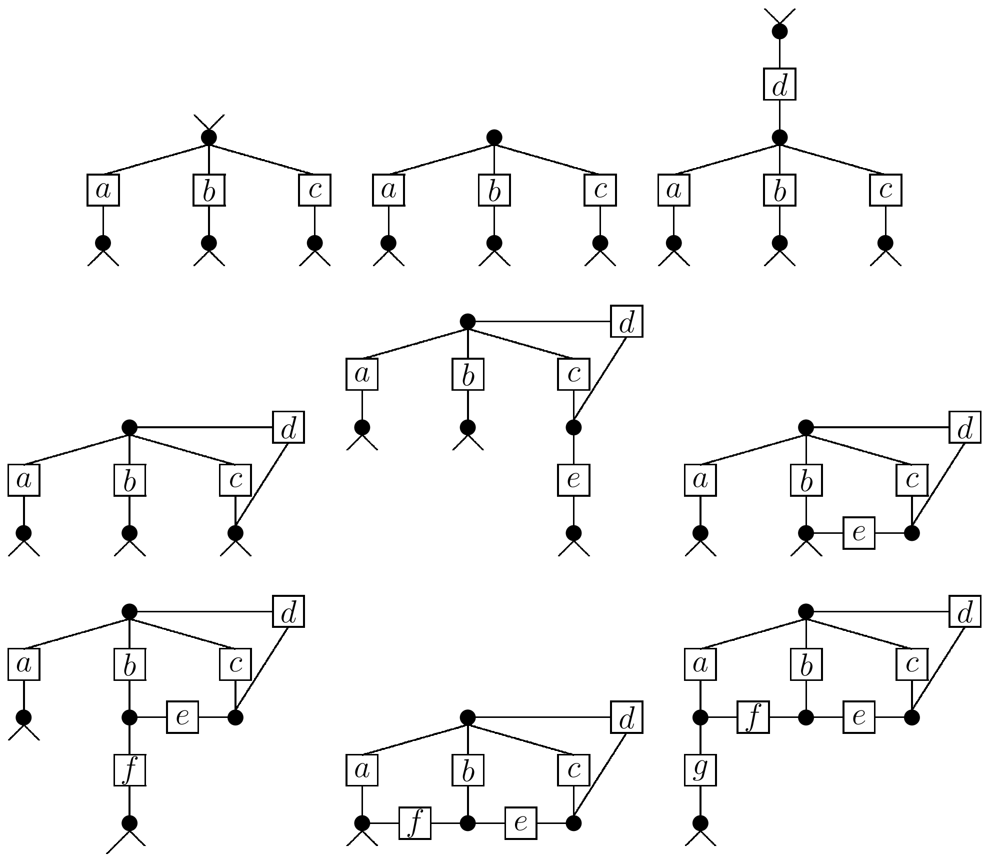

Figure 8

We have by now analyzed the structure of G far enough in order to describe a sufficiently small dominating set leading to the final contradiction. Let $V_{\geq 3}$ denote the set of vertices of degree at least 3 and let $n_{\geq 3} = |V_{\geq 3}|$. The graph $G[V \setminus V_{\geq 3}]$ is a collection of paths of order either 1 (mod 3) or 2 (mod 3).

Let $P_1, P_2, ..., P_s$ denote the set of vertices of the paths of order 1 (mod 3) and let $Q_1, Q_2, ..., Q_t$ denote the set of vertices of the paths of order 2 (mod 3).

By the above claims,

$$s + t \;\; \geq \;\; \frac{3n_{\geq 3}}{2} \quad \text{and} \quad s \leq n_{\geq 3}$$

which implies

$$t \;\; \geq \;\; \frac{n_{\geq 3}}{2} \quad \text{and} \quad \left(n_{\geq 3} - \frac{s}{3} - \frac{2t}{3} \right) \leq \frac{n_{\geq 3}}{3}.$$

For $1 \leq i \leq s$, the path $G[P_i]$ without its one or two endvertices has a dominating set D_i^P of cardinality $\frac{|P_i|-1}{3}$. For $1 \leq j \leq t$, the path $G[Q_j]$ without its two endvertices has a dominating set D_j^Q of cardinality $\frac{|Q_j|-2}{3}$.

Now the set

$$V_{\geq 3} \cup \bigcup_{i=1}^{s} D_i^P \cup \bigcup_{j=1}^{t} D_j^Q$$

is a dominating set of G and we obtain,

$$\begin{aligned}
\gamma &\leq n_{\geq 3} + \sum_{i=1}^{s} \left|D_i^P\right| + \sum_{j=1}^{t} \left|D_j^Q\right| \\
&= n_{\geq 3} + \sum_{i=1}^{s} \frac{|P_i| - 1}{3} + \sum_{j=1}^{t} \frac{|Q_j| - 2}{3} \\
&= \left(n_{\geq 3} - \frac{s}{3} - \frac{2t}{3}\right) + \sum_{i=1}^{s} \frac{|P_i|}{3} + \sum_{j=1}^{t} \frac{|Q_j|}{3} \\
&\leq \frac{n_{\geq 3}}{3} + \sum_{i=1}^{s} \frac{|P_i|}{3} + \sum_{j=1}^{t} \frac{|Q_j|}{3} \\
&= \frac{n}{3}.
\end{aligned}$$

This final contradiction completes the proof. $\square$

If the graph G arises from $l \geq 1$ disjoint copies of the cycle C_8 by choosing a set L of l vertices intersecting all these cycles and adding the edges of a tree on L, then $\gamma(G) = \frac{3}{8}n(G)$. Furthermore, $\gamma(C_{16}) = 6 = \frac{3 \cdot 16}{8}$. These examples show that Theorem 1 is best-possible for infinitely many graphs.

Note that $\gamma(C_{10}) = 4 > \frac{3n(C_{10})}{8}$. We believe that the assumption that the graphs in Theorem 1 do not contain cycles of lengths 10 might be replaced by the exclusion of finitely many exceptional graphs. For bipartite graphs we obtain the following.

Corollary 2 *If G is a bipartite graph of order n, minimum degree at least 2 and domination number γ that does not contain cycles of lengths 4 or 10, then $\gamma \leq \frac{3}{8}n$.*

We now proceed to bounds for the domination number of bipartite graphs of given minimum degree that are derived using the probabilistic method in a similar way as in the proof of (2) by Alon and Spencer [1]. In order to improve (2) we try to leverage the fact that the graph is bipartite. If for instance one of the partite sets is smaller than the other, then a minimum degree condition for the graph forces the average degree in the smaller partite set to be larger than the minimum degree which can be used to improve the estimate for the domination number.

Theorem 3 *If G is a bipartite graph with partite sets of cardinalities $n_A \leq n_B$, size m, minimum degree δ, maximum degree Δ and domination number γ, then*

$$\gamma \leq g_1 \leq g_2 \leq g_3 \leq g_4$$

where

$$\begin{aligned}
g_1 &= g_1(n_A, n_B, m, \delta, \Delta, a, b)\\
&= an_A + bn_B + (1-a)(1-b)^\delta \frac{\Delta n_A - m}{\Delta - \delta} + (1-a)(1-b)^\Delta \frac{m - \delta n_A}{\Delta - \delta}\\
&\quad +(1-a)^\delta(1-b)\frac{\Delta n_B - m}{\Delta - \delta} + (1-a)^\Delta(1-b)\frac{m-\delta n_B}{\Delta - \delta},\\
g_2 &= g_2(n_A, n_B, \delta, \Delta, a, b)\\
&= an_A + bn_B + (1-a)(1-b)^\delta \frac{\Delta n_A - \delta n_B}{\Delta - \delta} + (1-a)(1-b)^\Delta \frac{\delta n_B - \delta n_A}{\Delta - \delta}\\
&\quad +(1-a)^\delta(1-b)n_B,\\
g_3 &= g_3(n_A, n_B, \delta, a, b) = an_A + bn_B + (1-a)(1-b)^\delta n_A + (1-b)(1-a)^\delta n_B \text{ and}\\
g_4 &= g_4(n_A, n_B, \delta, a, b) = an_A + bn_B + e^{-a-\delta b} n_A + e^{-b-\delta a} n_B
\end{aligned}$$

for $0 \leq a, b \leq 1$.

Proof: Let the two partite sets be A and B with $n_A = |A|$ and $n_B = |B|$. We fix two probabilities $a \in [0,1]$ and $b \in [0,1]$ and select independently at random vertices from A with probability a and vertices from B with probability b. If $A' \subseteq A$ and $B' \subseteq B$ denote the sets of selected vertices, then $\mathbf{E}[|A'|] = an_A$ and $\mathbf{E}[|B'|] = bn_B$. If

$$A'' = \{u \in A \setminus A' \mid N_G(u) \cap B' = \emptyset\} \text{ and } B'' = \{u \in B \setminus B' \mid N_G(u) \cap A' = \emptyset\},$$

then $A' \cup A'' \cup B' \cup B''$ is a dominating set of G whose expected cardinality is an upper bound on γ and equals

$$\mathbf{E}[|A'|] + \mathbf{E}[|A''|] + \mathbf{E}[|B'|] + \mathbf{E}[|B''|] = an_A + bn_B + \mathbf{E}[|A''|] + \mathbf{E}[|B''|].$$

Now

$$\mathbf{E}[|A''|] = (1-a)\sum_{u\in A}(1-b)^{d_G(u)}.$$

Since $(1-b)^x$ is a convex function of x, $\delta \leq d_G(u) \leq \Delta$ for $u \in A$ and $\sum_{u\in A} d_G(u) = m$, the term $\sum_{u \in A}(1-b)^{d_G(u)}$ is at most

$$x(1-b)^\delta + (n_A - x)(1-b)^\Delta$$

where x is chosen as large as possible subject to the condition $\delta x + \Delta(n_A - x) \geq m$, i.e.

$$x = \frac{\Delta n_A - m}{\Delta - \delta}.$$

Therefore,

$$\mathbf{E}[|A''|] \leq (1-a)(1-b)^\delta \frac{\Delta n_A - m}{\Delta - \delta} + (1-a)(1-b)^\Delta \frac{m - \delta n_A}{\Delta - \delta}$$

and, by symmetry,

$$\mathbf{E}[|B''|] \leq (1-a)^{\delta}(1-b)\frac{\Delta n_B - m}{\Delta - \delta} + (1-a)^{\Delta}(1-b)\frac{m - \delta n_B}{\Delta - \delta}$$

which implies $\gamma \leq g_1$.

Since g_1 is decreasing in m and $m \geq \delta n_B$, we have $g_1 \leq g_2$. Since $(1-b)^{\delta} \geq (1-b)^{\Delta}$, we have $g_2 \leq g_3$ and, finally, since $1 + x \leq e^x$, we have $g_3 \leq g_4$ which completes the proof. $\square$

The problem of the bounds in Theorem 3 is that their evaluation involves the solution of the minimization problem of determining a and b such that g_i is smallest possible. The following observations are immediate: g_1 is smaller than g_2, if m is larger than δn_B and g_2 is smaller than g_3, if n_A is smaller than n_B. The problem remains to quantify these differences. The example in Figure 9 shows the functions g_1, g_2 and g_3 for $n_A = 200$, $n_B = 300$, $\delta = 20$, $\Delta = 100$, $m = 8000$, $a \in [0.13, 0.19]$ and $b \in [0.09, 0.15]$. In this case $\lfloor g_1(0.16, 0.11)\rfloor = 84$, $\lfloor g_2(0.16, 0.11)\rfloor = 87$ and $\lfloor g_3(0.16, 0.11)\rfloor = 89$.

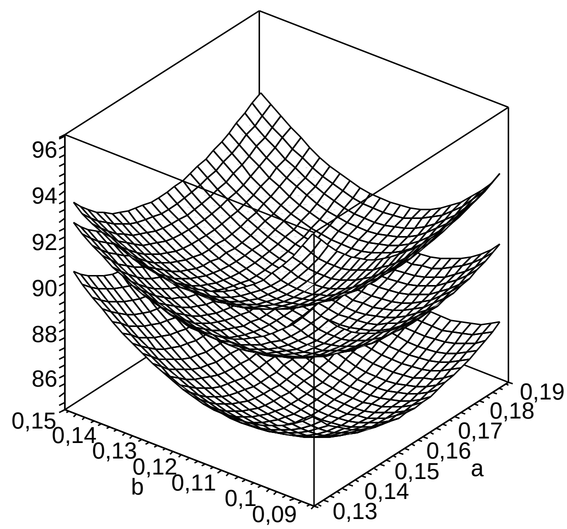

Figure 9

We will now show how to derive an explicit bound from

$$g_4 = g_4(a,b) = a n_A + b n_B + e^{-a-\delta b} n_A + e^{-b-\delta a} n_B$$

($n_A \leq n_B$ and $\delta \geq 2$ are considered to be fixed). Note that $g_4(a,b)$ is strictly convex as the sum of two linear and two strictly convex functions.

Let $n = n_A + n_B$ and $t = \frac{n_A}{n}$ with $0 < t \leq \frac{1}{2}$. We introduce two variables

$$x = e^{-a-\delta b} \text{ and } y = e^{-b-\delta a}.$$

Setting the partial derivatives $\frac{\partial}{\partial a}g_4(a,b) = n_A - n_A x - \delta n_B y$ and $\frac{\partial}{\partial b}g_4(a,b) = n_B - \delta n_A x - n_B y$ to zero, yields two simple linear equations for x and y which — for $\delta \geq 2$ — have the unique solution

$$x = x(t) = \frac{\delta n_B - n_A}{(\delta^2-1)n_A} = \frac{\delta(1-t)-t}{(\delta^2-1)t}$$

and

$$y = y(t) = \frac{\delta n_A - n_B}{(\delta^2-1)n_B} = \frac{\delta t-(1-t)}{(\delta^2-1)(1-t)}.$$

For the probabilities a and b this implies

$$a = a(t) = \frac{1}{\delta^2-1}(\ln x - \delta \ln y)$$

and

$$b = b(t) = \frac{1}{\delta^2-1}(\ln y - \delta \ln x).$$

In view of the strict convexity of g_4, the point

$$(a(t), b(t)) \in \mathbb{R}^2$$

is the unique global minimum of g_4. In order to guarantee that $a, b \in [0,1]$, we need $1 \leq \frac{x}{y^\delta} \leq e^{\delta^2-1}$ and $1 \leq \frac{y}{x^\delta} \leq e^{\delta^2-1}$. This can be ensured for instance by the symmetric conditions

$$e^{-\delta} \leq x \leq e^{-1} \text{ and } e^{-\delta} \leq y \leq e^{-1}.$$

In view of the explicit values given above for x and y in terms of t, this is equivalent to

$$t \;\geq\; \max\left\{\frac{\delta^2-1+e^\delta}{\delta^2-1+e^\delta(\delta+1)}, \frac{e\delta}{\delta^2-1+e(\delta+1)}\right\}$$

and

$$t \;\leq\; \min\left\{\frac{\delta^2-1+e}{\delta^2-1+e(\delta+1)}, \frac{e^\delta\delta}{\delta^2-1+e^\delta(\delta+1)}\right\}.$$

A simple calculation shows

$$\frac{\delta^2-1+e^\delta}{\delta^2-1+e^\delta(\delta+1)} \leq \frac{e\delta}{\delta^2-1+e(\delta+1)}$$

and

$$\frac{1}{2} \leq \frac{\delta^2-1+e}{\delta^2-1+e(\delta+1)} \leq \frac{e^\delta\delta}{\delta^2-1+e^\delta(\delta+1)}$$

and the condition on t simplifies to

$$\frac{e\delta}{\delta^2-1+e(\delta+1)} \leq t \leq \frac{1}{2}.$$

Note that

$$e^{-a-\delta b}n_A + e^{-b-\delta a}n_B = xn_A + yn_B = \frac{n}{\delta+1}.$$

Putting all this together we obtain the following.

Corollary 4 *If G is a bipartite graph of order n of minimum degree $\delta \geq 2$ with partite sets of cardinalities tn and $(1-t)n$ for some t with*

$$\frac{e\delta}{\delta^2 - 1 + e(\delta+1)} \leq t \leq \frac{1}{2},$$

then

$$\begin{aligned}
\gamma &\leq g_1(tn, (1-t)n, \delta(1-t)n, \delta, (1-t)n, a(t), b(t)) \\
&\leq g_2(tn, (1-t)n, \delta, (1-t)n, a(t), b(t)) \\
&\leq g_3(tn, (1-t)n, \delta, a(t), b(t)) \\
&\leq g_4(tn, (1-t)n, \delta, a(t), b(t)) \\
&= \frac{n}{\delta+1} + \frac{tn}{\delta^2-1}\left(\ln\left(\frac{\delta(1-t)-t}{(\delta^2-1)t}\right) - \delta \ln\left(\frac{\delta t-(1-t)}{(\delta^2-1)(1-t)}\right)\right) \\
&\quad + \frac{(1-t)n}{\delta^2-1}\left(\ln\left(\frac{\delta t-(1-t)}{(\delta^2-1)(1-t)}\right) - \delta \ln\left(\frac{\delta(1-t)-t}{(\delta^2-1)t}\right)\right) \\
&\leq g_4\left(tn, (1-t)n, \delta, \frac{\ln(\delta+1)}{\delta+1}, \frac{\ln(\delta+1)}{\delta+1}\right) \\
&= \left(\frac{1+\ln(\delta+1)}{\delta+1}\right) n.
\end{aligned}$$

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